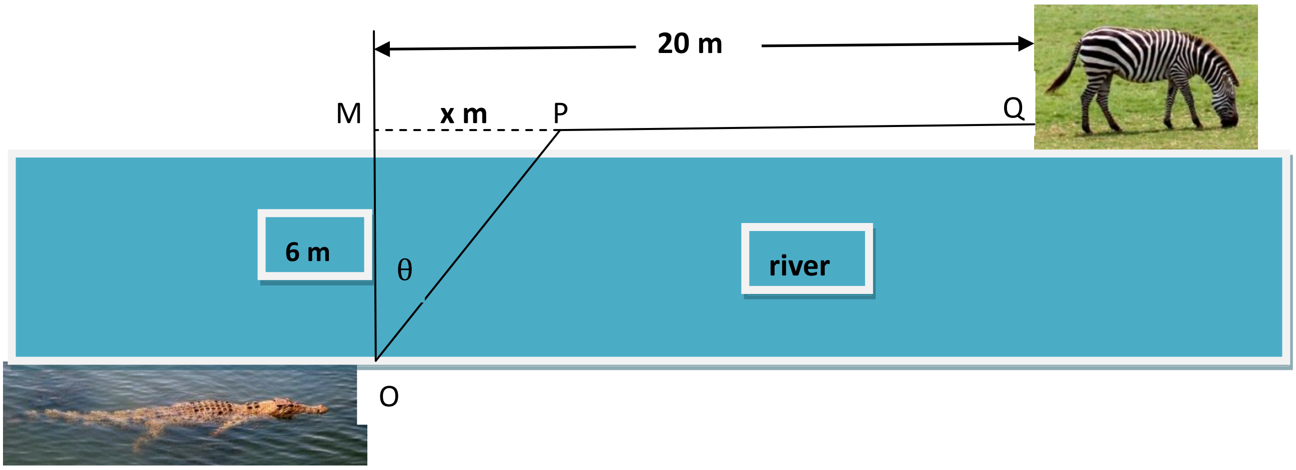
Crocodile catching its prey



A crocodile is stalking a zebra 20 m downstream on the opposite bank of the river, which is 6 m wide. The crocodile travels 0.25 m/s on land and 0.2 m/s in water. Let $T(x)$ be the time for the crocodile to reach the prey by swimming from point O to a particular point P, which is x m downstream on the other side of the river as shown in the diagram.

Find:

(a) $T(x)$ in terms of x .

(b) **(i)** Calculate the time if the crocodile does not travel on land.

(ii) Calculate the time if the crocodile swims the shortest time possible.

(c) Use the function in **(a)**, or otherwise, find the minimum time taken for the crocodile reach its prey.

(d) let $\angle MOP = \theta$

Let $t(\theta)$ be the time for the crocodile to reach the prey.

Find $t(\theta)$ and hence find the minimum time taken for the crocodile reach its prey.

Solution

(a) Total time = time travelled on land + time travelled in water

$$T(x) = \frac{\sqrt{36+x^2}}{0.2} + \frac{20-x}{0.25} = 5\sqrt{36+x^2} + 4(20-x) \quad (0 \le x \le 20)$$

(b) **(i)** $T(20) = 5\sqrt{36+20^2} + 4(20-20) = 104.4030650891055$

(ii) $T(0) = 5\sqrt{36+0^2} + 4(20-0) = 110$

(c) **Method 1A (Algebra)**

$T = 5\sqrt{36+x^2} + 4(20-x)$

$T - 80 + 4x = 5\sqrt{36+x^2}$

$(T - 80 + 4x)^2 = 25(36+x^2)$

$16x^2 + 8Tx - 640x + T^2 - 160T + 6400 = 900 + 25x^2$

$900 + 25\,x^2 - (16\,x^2 + 8\,T\,x - 640\,x + T^2 - 160\,T + 6400)$

$9\,x^2 \;+ (640 - 8\,T)\,x - T^2 + 160\,T - 5500 = 0$

$\Delta \geq 0 \;\Rightarrow (640 - 8\,T)^2 - 4(9)(-T^2 + 160\,T - 5500) \geq 0 \;\Rightarrow T^2 - 160T + 6076 \geq 0$

$\Rightarrow (T - 98)\,(T - 62) \geq 0 \Rightarrow T \leq 62$ or $T \geq 98$

By (a), $T \leq 62$ is not true.

Min. of $T(x) = 98$

$98 = 5\sqrt{36 + x^2} + 4(20 - x)$

$x = 8$

Method 1B (Algebra)

In order to make life easier, note that

$$T = 5\sqrt{36 + x^2} + 4(20 - x) = 5\sqrt{36 + x^2} - 4x + 80 = g(x) + 80$$

Since 80 is just a constant, we like to minimize $g(x)$. Then:

$4x + g = 5\sqrt{36 + x^2}$

$(4x + g)^2 = 25(36 + x^2)$

$16\,x^2 + 8\,g\,x + g^2 = 900 + 25x^2$

$9x^2 - 8gx + (900 - g^2) = 0$

Since x is real, we have:

$\Delta \geq 0 \Rightarrow 64g^2 - 4(9)(900 - g^2) \geq 0$

$\Rightarrow 16g^2 - (9)(900 - g^2) \geq 0 \Rightarrow 25g^2 - 8100 \geq 0 \Rightarrow g^2 - 324 \geq 0$

$\Rightarrow (g + 18)(g - 18) \geq 0 \Rightarrow g \leq -18$ or $g \geq 18$

By **(b)**, $g \leq -18 \Rightarrow T \leq 62$ is not true.

$\therefore$ Min. of $g = 18$ and Min. of $T(x) = 98$

$98 = 5\sqrt{36 + x^2} + 4(20 - x)$

$x = 8$

Method 2 (Calculus)

$T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$

$T'(x) = \frac{5\,x}{\sqrt{x^2+36}} - 4 = \frac{5\,x - 4\sqrt{x^2+36}}{\sqrt{x^2+36}}$

For critical values, $T'(x) = 0$

$\therefore\; 5\,x - 4\sqrt{x^2 + 36} = 0$

$\therefore\; x = \pm 8$

Since $x \geq 0$, $x = 8$

When x is slightly smaller than 8, $T'(x) < 0$.

When x is slightly bigger than 8, $T'(x) > 0$.

Hence Min. of $T(x) = T(8) = 5\sqrt{36 + 8^2} + 4(20 - 8) = 98$

(d) If we let $\angle MOP = \theta$

Let $t(\theta)$ be the time for the crocodile to reach the prey.

Then $MP = 6\tan\theta$, $OP = \frac{6}{\cos\theta}$, $PQ = 20 - 6\tan\theta$

$$t(\theta) = \frac{1}{0.2}\frac{6}{\cos\theta} + \frac{1}{0.25}(20 - 6\tan\theta) = \frac{30}{\cos\theta} + 4(20 - 6\tan\theta)$$

Since $0 \le PQ \le 20, 0 \le \tan\theta \le \frac{10}{3}$.

$$t(\theta) = 80 + \frac{30}{\cos\theta} - \frac{24\sin\theta}{\cos\theta} = 80 + \frac{30 - 24\sin\theta}{\cos\theta}$$

Method 3 (Calculus)

$$t(\theta) = 80 + \frac{30 - 24\sin\theta}{\cos\theta}$$

$$t'(\theta) = \frac{\cos\theta(-30\cos\theta) - (24 - 30\sin\theta)(-\sin\theta)}{\cos^2\theta} = \frac{24\sin\theta - 30}{\cos^2\theta}$$

$$t'(\theta) = 0 \Longrightarrow \sin\theta = \frac{24}{30} = \frac{4}{5}\ ,\ \cos\theta = \frac{3}{5}$$

Min. of t = $80 + \frac{30 - 24\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = 98$

Method 4 (Trigonometry)

$$t(\theta) = 80 + \frac{30 - 24\sin\theta}{\cos\theta} = 80 + y, \text{where} \quad y = \frac{30 - 24\sin\theta}{\cos\theta}$$

$$24\sin\theta + y\cos\theta = 30 \Longrightarrow \frac{24}{\sqrt{24^2 + y^2}}\sin\theta + \frac{y}{\sqrt{24^2 + y^2}}\cos\theta = \frac{30}{\sqrt{24^2 + y^2}}$$

Therefore, $\sin(\theta + \alpha) = \frac{30}{\sqrt{24^2 + y^2}}$ where $\tan\alpha = \frac{y}{24}$

Since $\sin(\theta + \alpha) \le 1$, $\frac{30}{\sqrt{24^2 + y^2}} \le 1 \Longrightarrow \sqrt{24^2 + y^2} \ge 30 \Longrightarrow y \ge 18$

Therefore $y_{min} = 18$

$t_{min} = 98$

Method 5 (Physics - Optics)

In optics, Fermat's principle or the principle of least time is the principle that the path taken between two points by a ray of light is the path that can be traversed in the least time.

Let a beam of light travelling at an incident angle $\theta = \angle MOP$ in a medium (water) travelling at a speed of 0.2 and the refracted ray PQ at an angle of $90°$ in another medium (land) travelling at a speed of 0.25.

Snell's law states that the ratio of the sines of the angles of incidence and refraction is equivalent to the ratio of phase velocities in the two media.

$$\frac{\sin\theta}{\sin 90°} = \frac{0.2}{0.25} \Longrightarrow \sin\theta = \frac{4}{5} \Longrightarrow \cos\theta = \frac{3}{5}$$

$$t(\theta) = 80 + \frac{30-24\sin\theta}{\cos\theta}$$

$$t_{\min} = 80 + \frac{30-24\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = 98$$

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